A Local Discontinuous Galerkin Method for Numerical Computation of Waveguide Eigenvalue Problems in Polar Coordinates

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Abstract — A numerical method for waveguide eigenvalue problems is presented using local discontinuous Galerkin (LDG) method based on polar coordinates. The method has the merit of avoiding geometrical triangulation errors on curved boundaries of the domain. High order accurate LDG method can be used with the proposed methodology because the proposed algorithm can be designed to cater for any order of accuracy. As an illustration, the formulation of the LDG scheme in polar coordinates is derived and several numerical examples are presented. Numerical results show that the proposed LDG method can solve waveguide eigenvalue problems accurately.

I. INTRODUCTION

The design of algorithms to compute, accurately, the waveguide eigenvalue of electromagnetics is a very topical challenge for researchers [1-3]. In practice, circular waveguide is very common and for the case being studied and report in this paper. The domain of waveguide eigenvalue problem is a disk with curvilinear boundary. It can be shown that with the use of traditional finite element method, geometrical errors are introduced when the curved boundary is approximated with piecewise line segments, thus the application of high order numerical method is limited. It has been reported in [4] that the observed convergence rate of the maximum norm, when using LDG method for the numerical solution of Laplace eigenvalue problem, is 2*k* when the *k*-th order finite element space is used. This implies that it is possible to obtain accurate numerical solution on a coarse mesh. In this paper a LDG method in polar coordinates is presented in order to obtain an accurate numerical solution of the waveguide eigenvalue problem which is defined in either circular domain or fan-shaped domain [5-6].

II. LDG SCHEME

Consider the following Helmholtz boundary value problem in polar coordinates:

$$\begin{aligned} &\Omega : -\nabla^2_{r,\theta} u - {k_c}^2 u = 0 \\ &\Gamma_1 : u = g \\ &\Gamma_2 : \frac{\partial u}{\partial n} = h \end{aligned} \quad , \tag{1}$$

where; Ω is bounded in R^2 ; $\partial\Omega = \Gamma_1 \bigcup \Gamma_2$; u is the unknown potential function; g, h are known functions; k_c is the waveguide eigenvalue to be solved numerically.

For illustration, define Ω as a bounded region in R^2 ; T is a triangulation of Ω ; h is the maximum side length of the triangulation. Define the *k*-th order discontinuous finite element space as:

$$V_h = \{u \in L^2(\Omega) : u|_K \in P^k(K), \forall K \in T\} . \tag{2}$$

Take any two neighboring elements K^+ and K^- from T , and the common side $\gamma = \partial K^+ \bigcap \partial K^-$, $\vec{n}^+$ and $\vec{n}^-$ are the unit outward normal vectors from the interior of K^+ and K^- at any point of γ , respectively. Let $w^\pm$ be the trace of w from the interior of $K^\pm$. Define $\{\{\cdot\}\}$, $[[\cdot]]$ as the respective average and jump of related function at $x \in \gamma$ *as*:

$$\begin{aligned} &\{\{u\}\} = (u^+ + u^-)/2, \quad \{\{\vec{q}\}\} = (\vec{q}^+ + \vec{q}^-)/2 \\ &[[u]] = u^+\vec{n}^+ + u^-\vec{n}^-, \quad [[q]] = \vec{q}^+\vec{n}^+ + \vec{q}^-\vec{n}^- \end{aligned} \quad . \tag{3}$$

Rewrite the partial differential equation in (1) as

$$\begin{cases} \vec{q} = [\dfrac{\partial u}{\partial r}, \dfrac{1}{r}\dfrac{\partial u}{\partial \theta}] \\ -\nabla_{r,\theta} \cdot \vec{q} - {k_c}^2 u = 0 \end{cases} . \tag{4}$$

Then the LDG scheme for (1) reads [7]

$$\begin{cases} \int_K \vec{q}_h \cdot \vec{w} dS = -\int_K u_h \nabla \cdot \vec{w} dS + \int_{\partial K} \hat{u}_h \vec{w} \cdot \vec{n} dl \\ \int_K \vec{q}_h \cdot \nabla v dS = \int_K {k_c}^2 uv dS + \int_{\partial K} v \hat{q}_h \cdot \vec{n} dl \end{cases} , \tag{5}$$

for any $(\vec{w}, v) \in {V_h}^2 \times V_h$; where $\hat{u}_h$ and $\hat{q}_h$ are the numerical fluxes which are defined by

$$\begin{aligned} &\hat{q}_h = \{\{\vec{q}_h\}\} - C_{11}[[u_h]] - \vec{C}_{12}[[\vec{q}_h]] \\ &\hat{u}_h = \{\{u_h\}\} + \vec{C}_{12}[[u_h]] \end{aligned} , \tag{6}$$

in the interior of the domain whereas

$$\begin{aligned} &\hat{q}_h = \begin{cases} \vec{q}_h{}^+ - C_{11}(u_h^+ - g)\vec{n}, \text{on } \Gamma_1 \\ h, \text{on } \Gamma_2 \end{cases} \\ &\hat{u}_h = \begin{cases} g, \text{on } \Gamma_1 \\ u_h^+, \text{on } \Gamma_2 \end{cases} \end{aligned} , \tag{7}$$

are defined on the boundary of the domain Ω ; where $C_{11} = O(1/h)$ and $\vec{C}_{12}$ are vectors in R^2 of length $1/2$.

In the following example, the waveguide eigenvalue problem is solved for both TE mode and TM mode. In the case of TM mode, the boundary condition is:

$$\partial\Omega : u = 0 . \tag{8}$$

In the TE mode, the boundary condition is:

$$\partial\Omega : \partial u / \partial n = 0 . \tag{9}$$

As the numerical flux $\hat{u}_h$ does not depend on $\vec{q}$, so after discretizing (1) using the LDG method, the unknowns about $\vec{q}$ can be eliminated from the derived linear algebraic equation. The generalized eigenvalue problem is therefore

$$Lu = {k_c}^2 Mu , \tag{10}$$

which can be solved by calling the solvers.

III. NUMERICAL RESULTS

In this section, the numerical results of two examples are illustrated using the 4-th order LDG method. The computational mesh numbers in (r,θ) are given in the titles of the tables below. It can be seen that accurate eigenvelue can be obtained using high order LDG method with a coarse mesh.

Example I is a circular waveguide eigenvalue problem in both TM mode and TE mode. In the case being studied, the exact eigenvalue can be obtained by finding the zeros of the Bessel function of the first kind.

TABLE I
EIGEN VALUES OF EXAMPLE I IN TM MODE (5×10)

#	Numerical solution	Exact solution
1	2.40482555769731	2.40482555769577
2	3.83170602101311	3.83170597020751
3	5.13562612644477	5.13562230184068
4	5.52007811318256	5.52007811028631

TABLE II
EIGEN VALUES OF EXAMPLE I IN TE MODE (5×10)

#	Numerical solution	Exact solution
1	1.84118378883568	1.84118378134066
2	3.05423806089708	3.05423692822714
3	3.83170597034596	3.83170597020751
4	4.20121034061308	4.20118894121053

Examples II and III are sector waveguide eigenvalue problems in both TM mode and TE mode, where the central angle of the sector are taken to be $\pi/2$ and $3\pi/2$, respectively. As the analytical exact eigenvalues are not easy to obtain in both cases, hence the numerical solutions in [5] are used for reference.

TABLE III
EIGEN VALUES OF EXAMPLES II, III IN TM MODE (5×10)

#	Numerical solution	Reference solution [5]
Central angle = $\pi/2$		
1	5.13562236681274	5.1413
2	7.58834446643681	7.6027
3	8.41724661329691	8.4758
4	9.93612744395106	9.9755
Central angle = $3\pi/2$		
1	3.37487023849970	3.3755
2	4.27533749576592	4.2772
3	5.13562341195871	5.1411
4	5.97014468913302	5.9864

TABLE IV
EIGEN VALUES OF EXAMPLES II, III IN TE MODE (5×5)

#	Numerical solution	Reference solution [5]
Central angle = $\pi/2$		
1	3.05423697056096	3.0549
2	3.83170597041761	3.8386
3	5.31755803154267	5.3276
4	6.70613666189395	6.7142
Central angle = $3\pi/2$		
1	1.40089061563629	1.4012
2	2.25775440817062	2.2613
3	3.05423716978195	3.0827
4	3.82322572156638	3.8326

Fig. 1 shows the computed eigenvectors corresponding to the eigenvalues of the TM mode of Example II (multiple eigenvectors are plotted separately).

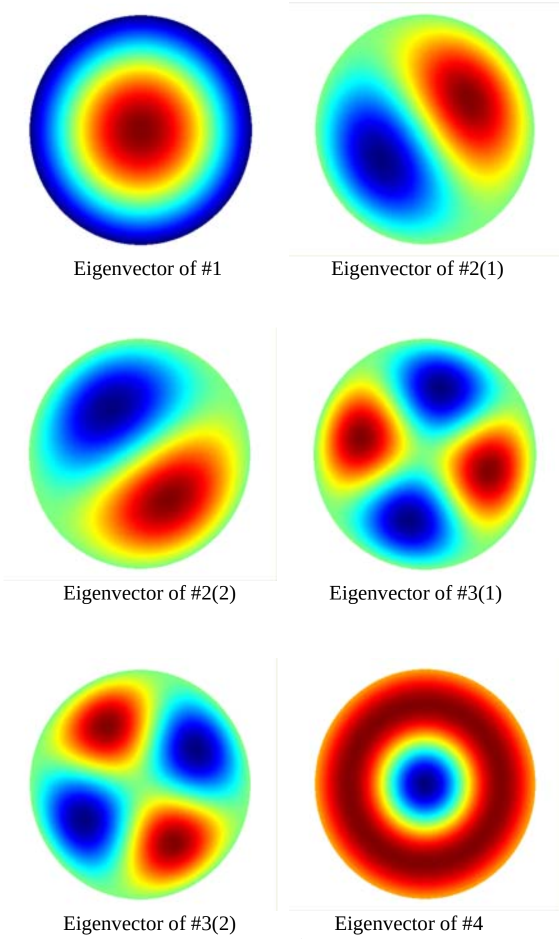

Fig. 1. First 6 eigenvectors of Example I in TM mode.

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